Statistical mechanics of self-driven Carnot cycles

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The spontaneous generation and finite-amplitude saturation of sound, in a traveling-wave thermoacoustic engine, are derived as properties of a second-order phase transition. It has previously been argued that this dynamical phase transition, called ''onset,'' has an equivalent equilibrium representation, but the saturation mechanism and scaling were not computed. In this work, the sound modes implementing the engine cycle are coarse-grained and statistically averaged, in a partition function derived from microscopic dynamics on criteria of scale invariance. Self-amplification performed by the engine cycle is introduced through higher-order modal interactions. Stationary points and fluctuations of the resulting phenomenological Lagrangian are analyzed and related to background dynamical currents. The scaling of the stable sound amplitude near the critical point is derived and shown to arise universally from the interaction of finite-temperature disorder, with the order induced by self-amplification. [S1063-651X(99)03210-9]

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I. INTRODUCTION

Self-organizing engines form a pervasive and fascinating class of objects, but one about which little of a general nature has been understood. These are systems in which at least part of the work generated by an engine cycle reinforces or amplifies the event sequence implementing the cycle. Through this amplification, the cycle can be spontaneously generated from noise, so these engines are self-starting. Reinforcement of a given cycle is also typically accompanied by the suppression of other modes of fluctuation, so they are self-organized. Examples of such systems include weather patterns such as tornadoes and hurricanes, autocatalytic chemical networks, and all living things.

Many empirical features of these engines suggest that they may be controlled by the interplay of dynamics and statistics, and in particular that the self-starting transition may be mathematically a phase transition. While an engine cycle can self-generate from noise, its amplification does not persist indefinitely; the driven cycle tends to saturate at some stable rate of heat transport dependent on how strongly the system is driven. The absence of an intrinsic scale for this saturation, together with the typical existence of threshold values for self-generation to proceed at all, are characteristics of the order parameter near a second-order critical point.

On the other hand, the long-range order thus created is of an event sequence *in time*, and the transport processes responsible are explicitly nonequilibrium. Further, the engine cycle, once selected, can often be described classically. The interesting question raised by these systems, then, is whether statistical disorder can select such dynamical backgrounds, and stabilize them even at an apparently classical level.

One way to shed light on the general problem is to thoroughly analyze a representative case. The self-starting thermoacoustic engines [1–4] are ideal examples of the generation of dynamic structure, because they demonstrate all of the above properties, but offer a few important simplifications for analysis.

Thermoacoustic engines are resonators that spontaneously generate a stable sound wave when driven by a thermal gradient exceeding some threshold value [2]. The amplitude of the generated mode, like a phase-transition order parameter, is a nonanalytic function of the temperature gradient driving the engine, which serves as an effective coupling. For gradients below critical, sound fluctuations do not organize, and the equilibrium state of the engine is quiescent. For gradients above critical, a fundamental resonator mode of arbitrary phase (arbitrary zero of time) spontaneously develops a stable nonzero amplitude, which grows monotonically in the coupling, with initially infinite slope at the critical point [5]. This crossing of the threshold for spontaneous generation of sound is called ''onset.'' The existence, above onset, of a stable phase for the running cycle is tantamount to vanishing of the degenerate, orthogonal phase, and the arbitrary but definite zero of time thus selected spontaneously breaks time translation, a symmetry of the underlying dynamics also expressed by the quiescent equilibrium. Onset thus has all the mathematical signatures usually identified with a phase transition.

At the same time, the engine cycle can be treated classically [6], and the exponential growth of sound away from an artificially quenched (supercooled) quiescent state above onset is well described this way [7]. The linear gain equation describing growth cannot predict saturation, however, so whether it is statistical or deterministic in origin is not known. It *is* known that saturation at small amplitudes is not associated with either reduced transport [2] or obvious harmonic generation or chaos [7], making a statistical mechanism plausible.

The simplification afforded by these engines is that their entire ''working machinery'' can consist of the acoustic resonance and conduction of an ideal gas. All excitations, whether dynamical modes that can be treated classically or thermal modes that must sum statistically, are thus phonons. The absence of a fundamental distinction between the two provides a bootstrap approach—scale-invariant treatment of *all* phonons—to inferring the statistical sum for the whole engine from that of the underlying finite-temperature gas.

Such scale invariance has been used [8] to assign an analytic structure to the effective action giving classical engine dynamics—based on its assumed embedding in a finite-temperature partition function. A conformal symmetry, mapping time and temperature scales of ideal gas states related by adiabatic transformations, was then used to show that a reversible thermoacoustic engine is equivalent to a system in apparent thermal equilibrium. The two together led to a proof of Carnot's theorem as conservation of a Noether current, requiring only finite temperature, analyticity, and broken symmetry [9]. The emergence of a nontrivial conservation law from these assumptions thus furnished evidence that residual statistical properties may remain important at the classical level.

Only the analytic properties of the classical effective action—hence the symmetries and conservation laws of single engine cycles—were treated in Ref. [8]. The completion of the statistical sum to include macroscopic modes was not carried through, so spontaneous symmetry breaking, or a statistical origin for the saturation amplitude, could not be shown. Those results will be derived here. Because much of the groundwork for describing the example engine was laid in the previous work, forms for relevant effective actions will be cited there, and only reproduced when specifically used in the present calculations. The derivation is arranged as follows.

The explicit structure of the partition function describing the engine at the level of ideal gas dynamics, not previously needed for the treatment of individual engine cycles, will first be presented in Sec. II. It shows formally how the microscopic thermal sum, to be defined consistently, must include macroscopic fluctuations as well. It also shows how the conformal symmetry of the effective dynamical action in Ref. [8] leads to an equivalent equilibrium representation.

The local fluctuations that make up individual engine configurations are microscopic degrees of freedom, though, with respect to the modal amplitudes that appear to have critical behavior, so the form of the microscopic effective action is not directly useful. Therefore, the formal construct of Sec. II will be coarse-grained in Sec. III, and the local gas constraints replaced by a phenomenological constraint representing the Carnot efficiency of a reversible cycle (the only kind whose dynamics are encoded in an analytic effective action). The resulting phenomenological Lagrangian is the most general, consistent with the symmetries of weakly perturbed phonons and the definition of the system as an engine. As expected from the formal arguments of Ref. [8] and Sec. II, it has an analytic continuation leading to an equivalent, finite-temperature Euclidean field theory in apparent thermal equilibrium. This form will then be scaled, and the cutoff on high-frequency modes implicitly renormalized, to yield the effective field theory in which the stationary points of the action actually correspond to classical configurations of the system.

The only free parameter in this derivation is the value of an effective coupling, representing the strength with which the Carnot cycle feeds its own growth. It is not clear how such a value can either be derived from microphysics or inferred from direct measurements, so only the scaling of the coupling with driving gradient is derived.

The general form of the effective action is then expanded in mean-field theory in Sec. IV, and shown to lead to a nonzero current background that has an immediate interpretation in terms of broken time-translation symmetry in the dynamical sector. An analysis of fluctuations about stationary points predicts a universal scaling for the amplitude of driven sound in a neighborhood of the critical point, which is the same as that for the magnetization in a Ginzburg-Landau ferromagnet. The way the effective coupling appears in this formula gives a prescription for assigning it a value, if this scaling regime can be found experimentally.

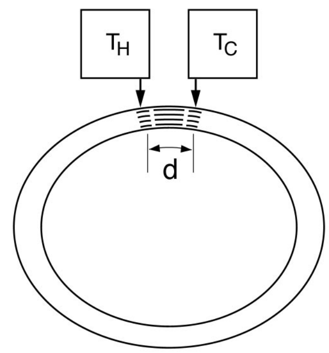

FIG. 1. Schematic of the traveling-wave engine.

The generality of these results deserves comment. There are two fundamental classes of thermoacoustic engines currently known: a standing-wave (SW) engine with intrinsically irreversible dynamics [1] and a finite critical driving gradient, and a traveling-wave (TW) engine that is reversible in idealized limits [3,4], with a critical point at zero gradient. The reversible TW engine is the system considered in Ref. [8] and here. The time-translation-breaking properties of the two onset transitions are similar, but the natures of the coupling driving the gas, and the time-reversal symmetries of the cycles, are different.

The main conclusion of this derivation—that the local exponential growth of self-driven cycles saturates due to finite-temperature statistics—will clearly follow from general features of the phenomenological Lagrangian, not peculiar to the engine used as an example. However, the particular phase structure may be different for different cycles, and (regrettably) it will be left as an open question whether the TW and SW onset transitions are of the same universality class.

II. STATISTICAL REPRESENTATION OF THE ACOUSTIC STIRLING ENGINE

A. The self-consistency bootstrap

The system considered in Ref. [8] and here is a traveling-wave engine introduced by Ceperley [3], and shown schematically in Fig. 1. It consists of an annular resonator filled with an ideal gas, a stack of closely spaced, parallel plates in the flow stream of the gas, and two thermal reservoirs coupled to heat exchangers at opposite ends of the stack. The stack length will be called d, and the resonator length $L \gg d$. The resonator is assumed to admit one-dimensional flow, and the function of the stack is to enforce a constraint of zero temperature fluctuation at each position along its length. Under this constraint, the pressure-velocity phasing of traveling waves implements a Stirling cycle, which has Carnot efficiency in the limit of idealized stack coupling.

The important advantage of this engine as an example, over the more familiar and easily realized standing-wave version, is that thermally induced irreversibility is not intrinsically necessary to perform the cycle. The stack spacing is assumed much smaller than thermal boundary layers, so the fixed-temperature boundary condition it imposes on the gas in the idealized limit may be modeled with a ''perfect conductivity,'' which simply represents the classic approximation of reversible isothermal heat exchange. All other evolution of the gas in the engine may be idealized as adiabatic. One consequence is that the idealized cycle, being reversible, must have Carnot efficiency. Therefore, the classical gain [3,8] is nonzero at any driving gradient, so the critical point for onset is classically expected to lie at zero gradient. Formation of order is still nonanalytic, because the driven mode is whichever traveling wave propagates in the direction of the stack gradient, so the two cases of positive and negative gradient couple to orthogonal modes (which may be regarded as dual order/disorder parameters).

The other consequence is that all necessary engine dynamics, even at the microscale, is consistent with derivation from an effective action. The reversible isothermal heat transfer may be implemented with Lagrange-multiplier constraints, while adiabatic dynamics follows from an appropriate free field theory. Thus, the explicit form of the effective action, at each scale of averaging, may be expected to represent the essential dynamics of the engine, without requiring augmentation by irreversible terms that cannot be absorbed by renormalization into the action itself.

Construction of the partition function for the whole engine begins by specification of the bare partition function for free phonons, and then proceeds by successive averaging of higher-frequency modes, to produce coarse-grained effective actions for the remaining degrees of freedom. The forms used for this construction in the rest of the subsection are drawn from Ref. [8]. The finite-temperature field theory of a free phonon gas is simplest in Lagrangian coordinates: x, the instantaneous position of a given parcel of gas in the resonator, is an embedding of the Lagrangian ''accumulated mass'' coordinate m into physical space. m is periodic modulo M, the total mass of gas.

At fixed temperature and no stack coupling, the isothermal partition function is defined [10] as

$$Z^{\text{iso}}=\int \mathcal{D}x e^{-S_E^0[x]/\hbar}, \tag{1}$$

where $\hbar$ is Planck's constant, and

$$S_E^0[x]\equiv \oint d\tau \oint dm \frac{1}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 \tag{2}$$

is the so-called Euclidean action for free massless bosons (phonons) on the domain m [11].

τ has units of time, and the system has temperature T_0 when τ is made periodiodic with $\oint d\tau=\hbar/k_B T_0$, where k_B is Boltzmann's constant. Low-frequency correlations of ''dynamical'' fields $\overline{x}$ can be studied by splitting $x=\overline{x}+x'$, and averaging over ''thermal'' fluctuations x':

$$Z^{\text{iso}}=\int \mathcal{D}\overline{x}\int \mathcal{D}x'\, e^{-S_E^0[\overline{x}+x']/\hbar}\equiv \int \mathcal{D}\overline{x}\, e^{-S_E[\overline{x};T_0]/\hbar}. \tag{3}$$

''Classical'' dynamical correlations are *conditional* probabilities for configurations $\overline{x}$, where the condition collapses some part of the integral $\int \mathcal{D}\overline{x}$, and $S_E[\overline{x};T_0]$ can be used as a classical action to give δ-function *conditioned* probabilities for the rest of the configuration (classical field configurations from initial conditions). The form of $S_E[\overline{x};T_0]$, constrained by the requirement that it produce the correct equations of motion for isothermal sound, is given as Eq. (5) of Ref. [8] (with T set to T_0).

It was further shown in Ref. [8], however, that $S_E[\overline{x};T_0]$ in Eq. (3) can be regarded as $S_E[\overline{x};T]$, with the insertion of $1=\int \mathcal{D}T\ \delta[T-T_0]$ in the functional integral (3), to represent the constraint of isothermal sound explicitly. The generalization to adiabatic dynamics comes by simply removing $\delta[T-T_0]$, to make T a fluctuating auxiliary field. The constraint it enforces then becomes conservation of comoving entropy.

For slow fluctuations (dynamics approaching local equilibrium everywhere), T may be taken as the conformal factor of a coordinate transformation in which τ, as well as $\overline{x}$, becomes an embedding field defined on a manifold of *two* Lagrangian coordinates (ζ,m). ζ is an affine parameter around $\oint d\tau$, as m is around $\oint dx$. By setting $\oint d\zeta\equiv T_0$, an arbitrary reference temperature, the resulting adiabatic partition function is given the form of equilibrium:

$$Z^{\text{ad}}=\int \mathcal{D}\tau\,\mathcal{D}\overline{x}\, e^{-S_E[\tau,\overline{x}]/\hbar}, \tag{4}$$

with S_E given as Eq. (14) of Ref. [8].

Finally, the stack coupling is introduced by means of Lagrange multipliers λ in a constraint action S_C, which enforce nonfluctuating temperature at positions along the stack by fixing the conformal factor. In the partition function, the λ become auxiliary fields implementing δ functionals of the constraints, and the formal equilibrium partition function for the whole engine becomes

$$Z^{\text{engine}}=\int \mathcal{D}\tau\,\mathcal{D}\overline{x}\,\mathcal{D}\lambda\, e^{-(S_E+S_C)/\hbar}. \tag{5}$$

S_C is given as Eq. (15) of Ref. [8].

The formal existence of the representation (5) is the starting point for constructing the phenomenological description of the TW engine. The detailed structure of S_E and S_C, beyond the free terms of Eq. (2), is not needed, because higher modes than the fundamentals ones are not of interest to describe the basic onset transition. The local gas dynamics also does not appear at the level of coarse-graining describing symmetry breaking. Only the independent fundamental mode amplitudes need be kept as dynamical variables, and only the cycle-averaged gain relation proceeding from analysis of the local gas dynamics is needed to define their interactions.

B. Coarse-graining and the phenomenological Lagrangian

The detailed structure of the phenomenological Lagrangian for symmetry breaking generally does not resemble that for its underlying dynamics [12]. In particular, the explicit conformal factor for embedding τ and the local-conduction auxiliary fields λ in Eq. (5) need not appear at all. Rather, the effective theory is defined by symmetries, and the order of terms appearing in it is determined by the limit it must approach as a free theory.

For the TW engine, the fact that pressure and density fluctuations remain in phase while performing the Stirling cycle leads to particularly simple kinematics. For a short stack, as is usually assumed [3,4,8], temperature and density fluctuations over most of the resonator (all but a length $d/L \ll 1$, where temperature does not fluctuate) remain in phase, and velocity is determined by the equation of continuity, so a single real variable describes the configuration space. A dimensionless displacement potential ϕ (variously normalized) will be used here.

The engine partition function must reduce to that for free sound at zero stack coupling, because the gain relation is a small perturbation on the resonance condition that makes modal amplitudes well defined. Also, because ∂_ζ and ∂_τ, or ∂_m and ∂_x, are the same at leading order in small fluctuations, and the phenomenological Lagrangian is expanded to leading nontrivial order in derivatives, ∂_τ and ∂_x may be used to define the mode bases.

With these conventions, the two independent spatial basis functions for arbitrary fundamental-mode pressure fluctuations are $\hat{e}_1 \equiv \cos(k_0 x)$ and $\hat{e}_2 \equiv \sin(k_0 x)$, where $k_0 \equiv \omega_0/\bar{c}$, ω_0 is the resonance frequency, and $\bar{c}$ the mean sound speed. In the starting microscopic theory, an arbitrary ''bare'' phonon configuration may be expanded in real functions of time $\phi_{R,B}$, $\phi_{I,B}$ as $\phi_B = \phi_{R,B}\hat{e}_1 + \phi_{I,B}\hat{e}_2$. It is intuitive to replace $\hat{e}^1 \to 1$, $\hat{e}_2 \to i$, and let $\phi_B \to \phi_{R,B} + i\phi_{I,B}$ be a complex-valued scalar, so that $e^{\pm i\omega_0 t}$ are, respectively, analytic and antianalytic traveling waves.

The partition function for free sound is most easily constructed by analytic continuation from the generating functional of dynamic correlations. The wave equation for $\phi_{R,B}$, $\phi_{I,B}$ [as classical fields, up to a scale factor not specified by the free theory, at the internal level of Eq. (3), or operators in the full average], is

$$(\partial_t^2 + \omega_0^2)\begin{bmatrix}\phi_{R,B}\\ \phi_{I,B}\end{bmatrix} = 0. \tag{6}$$

Green's functions for the operator (6) are generated by

$$\zeta^{\text{free}} \equiv \int \mathcal{D}\phi_{R,B}\,\mathcal{D}\phi_{I,B}\, e^{-iS^{\text{free}}/\hbar}, \tag{7}$$

where the combined requirement to recover Eq. (6) by variation, and for the kinetic term to continue to a mode expansion of Eq. (2) in small fluctuations, $\delta x/L \equiv L\partial_x\phi_B/2\pi$, specifies

$$\frac{S^{\text{free}}}{\hbar} \equiv \frac{1}{\omega_R}\int \frac{dt}{2}\{(\partial_t\phi_{R,B})^2 + (\partial_t\phi_{I,B})^2 - \omega_0^2(\phi_{R,B}^2 + \phi_{I,B}^2)\}. \tag{8}$$

$\omega_R \equiv \hbar/L^2 M$ defines the fundamental reference frequency of the bare theory.

III. ANALYTIC CONTINUATION AND THE EQUILIBRIUM EFFECTIVE ACTION

The finite-temperature partition function for free sound is obtained from the generating functional (7), by considering all correlations as analytic functions of the complex variable $z = t + i\tau$ [10]. Rotating the contour along which the functional integral, and its correlations, are evaluated, gives the continuation $t \to i\tau$, with corresponding continuation of derivatives $\partial_t \to \partial_z \to -i\partial_\tau$. The continuation of Eq. (6) gives the Euclidean equations of motion:

$$(-\partial_\tau^2 + \omega_0^2)\begin{bmatrix}\phi_{R,B}\\ \phi_{I,B}\end{bmatrix} = 0. \tag{9}$$

The generating functional of temporal correlations continues to the finite temperature partition function, $\zeta^{\text{free}} \to Z^{\text{free}}$, in which the dynamic action continues to the so-called ''Euclidean action,'' $-iS^{\text{free}} \to -S_E^{\text{free}}$. Denoting $\partial_\tau() \equiv (\dot{\ })$, S_E^{free} is given by

$$\frac{S_E^{\text{free}}}{\hbar} \equiv \frac{1}{\omega_R}\oint \frac{d\tau}{2}\{\dot{\phi}_{R,B}^2 + \dot{\phi}_{I,B}^2 + \omega_0^2(\phi_{R,B}^2 + \phi_{I,B}^2)\}. \tag{10}$$

Equation (10) is just the Fourier transform of S_E in Eq. (4) with respect to m, projected onto the lowest modes, as required for scale-invariant *definition* of the partition function defining the free-phonon theory.

The form of the perturbations that must be added to the coarse-grained free action (10) can be inferred from the modal gain analysis in Appendix B of Ref. [8], which in turn follows from $(S_E + S_C)$ in Eq. (5). The idealized acoustic Stirling engine has Carnot efficiency and no load, so all energy from the TW cycle drives an in-phase amplification of the working mode. Since the stored energy is proportional to the intensity $(\phi_{R,B}^2 + \phi_{I,B}^2)$ and the energy flux comes entirely from the TW current $(\phi_{R,B}\partial_t\phi_{I,B} - \phi_{I,B}\partial_t\phi_{R,B})$, the gain equation (B17) of Ref. [8] can be expressed as

$$\partial_t(\phi_{R,B}^2 + \phi_{I,B}^2) = \frac{1}{Q}(\phi_{R,B}\partial_t\phi_{I,B} - \phi_{I,B}\partial_t\phi_{R,B}), \tag{11}$$

where

$$\frac{1}{Q} = \frac{1}{2\pi\gamma}\frac{\Delta T}{T}. \tag{12}$$

$\Delta T/T \equiv (T_H - T_C)/T_C$ in Fig. 1 is positive for gradients in the analytic TW direction, and $\gamma \equiv c_P/c_V$ is the ratio of isobaric to isochoric specific heats of the ideal gas. Since TW currents from the fundamental mode scale as ω_0, $1/Q$ is the fractional growth in energy per cycle passing the stack.

As it must if Eq. (5) is to have a coarse-grained form, the temporal energy conservation relation (11) immediately continues to the Euclidean section:

$$(\phi_{R,B}\dot{\phi}_{R,B}+\phi_{I,B}\dot{\phi}_{I,B})+\frac{1}{2Q}(\phi_{I,B}\dot{\phi}_{R,B}-\phi_{R,B}\dot{\phi}_{I,B})=0. \tag{13}$$

Introducing the notation $\tan\chi\equiv 1/2Q$, $c\equiv\cos\chi$, $s\equiv\sin\chi$, Eq. (13) can be compactly represented as vanishing of a scalar constraint function of the bare fields,

$$\mathcal{C}_B\equiv\begin{bmatrix}\phi_{R,B} & \phi_{I,B}\end{bmatrix}\begin{bmatrix}c & -s\\ s & c\end{bmatrix}\begin{bmatrix}\dot{\phi}_{R,B}\\ \dot{\phi}_{I,B}\end{bmatrix}=0, \tag{14}$$

which can be imposed on statistical correlations by insertion of the δ functional,

$$\delta[\mathcal{C}_B]=\mathcal{N}\int\mathcal{D}\lambda\, e^{i\oint d\tau\,\lambda\mathcal{C}_B} \tag{15}$$

($\mathcal{N}$ is a normalization constant), giving a candidate for the symmetry-breaking partition function:

$$Z=\int\mathcal{D}\phi_{R,B}\,\mathcal{D}\phi_{I,B}\,e^{-S_E^{\mathrm{free}}/\hbar}\,\delta[\mathcal{C}_B]. \tag{16}$$

However, a more general relation, which also does not involve an auxiliary field, is obtained by regulating the δ functional to a finite width about $\mathcal{C}_B=0$, in view of the finite thermal coupling between the working fluid and the stack. It is also only physically motivated to constrain low-frequency components of $\mathcal{C}_B$, because the Carnot relation is only defined over cycle averages. Both forms of regulation are accomplished by taking

$$\delta[\mathcal{C}_B]\to\delta_{g_B}[\overline{\mathcal{C}_B}]\equiv\mathcal{N}(g_B,\overline{\omega})\times\int\mathcal{D}\lambda\, e^{-(\omega_R/2g_B)\oint d\tau\lambda(1-g_B\partial_\tau^2/\overline{\omega}^2)\lambda}\,e^{i\oint d\tau\,\lambda\mathcal{C}_B}, \tag{17}$$

for a large ''bare coupling'' g_B. Components of λ at frequencies greater than $\overline{\omega}$ are suppressed in $\int\mathcal{D}\lambda$, so only the average of $\mathcal{C}_B$ over a time $\sim 2\pi/\overline{\omega}$, denoted $\overline{\mathcal{C}_B}$, is constrained to vanish. Physically, one expects that $\overline{\omega}\sim\omega_0/Q$, but the detailed value need not be specified here, because in mean-field-theory (MFT) calculations, it will only appear together with g_B in an effective coupling, which is chosen phenomenologically from the dynamical equations. Completing the square in Eq. (17), and defining the normalization $\mathcal{N}(g_B,\overline{\omega})$ for convenience, gives

$$\delta_{g_B}[\overline{\mathcal{C}_B}]\approx e^{-(g_B/2\omega_R)\oint d\tau\overline{\mathcal{C}}_B^2}\equiv e^{-S_E^{\mathrm{int}}/\hbar}. \tag{18}$$

The case $\overline{\omega}\to\infty$ recovers the instantaneous constraint: $\delta_{g_B}[\overline{\mathcal{C}_B}]\to\delta_{g_B}[\mathcal{C}_B]$.

Adding a matrix-valued source term,

$$\frac{S_E^{\mathrm{src}}}{\hbar}=-\frac{1}{\omega_R}\oint d\tau\begin{bmatrix}\phi_{R,B} & \phi_{I,B}\end{bmatrix}J\begin{bmatrix}\dot{\phi}_{R,B}\\ \dot{\phi}_{I,B}\end{bmatrix}, \tag{19}$$

to probe expectation values of growth and transport currents, the Carnot-constrained partition function is then defined in terms of the free Euclidean theory and the regulated constraint:

$$Z[J]=\int\mathcal{D}\phi_{R,B}\,\mathcal{D}\phi_{I,B}\,e^{-S_E[\phi_{R,B},\phi_{I,B};J]/\hbar}, \tag{20}$$

with $S_E=S_E^{\mathrm{free}}+S_E^{\mathrm{int}}+S_E^{\mathrm{src}}$. It is shown in Appendix A that, though only the averaged constraint $\overline{\mathcal{C}_B}$ is physically required to vanish, MFT results can be computed by substituting the more convenient, time-local constraint $\mathcal{C}_B$ for $\overline{\mathcal{C}_B}$ with a compensating shift of the coupling, $g_B\to\overline{g_B}\equiv g_B\overline{\omega}/\Lambda$, where Λ is a high-frequency cutoff. The resulting action in Eq. (20) then becomes

$$\begin{aligned}\frac{S_E}{\hbar}=\frac{1}{\omega_R}\oint\frac{d\tau}{2}\Bigg\{&\begin{bmatrix}\dot{\phi}_{R,B} & \dot{\phi}_{I,B}\end{bmatrix}\begin{bmatrix}\dot{\phi}_{R,B}\\ \dot{\phi}_{I,B}\end{bmatrix}\\ &+\omega_0^2\begin{bmatrix}\phi_{R,B} & \phi_{I,B}\end{bmatrix}\begin{bmatrix}\phi_{R,B}\\ \phi_{I,B}\end{bmatrix}\\ &+\overline{g_B}\left(\begin{bmatrix}\phi_{R,B} & \phi_{I,B}\end{bmatrix}\begin{bmatrix}c & -s\\ s & c\end{bmatrix}\begin{bmatrix}\dot{\phi}_{R,B}\\ \dot{\phi}_{I,B}\end{bmatrix}\right)^2\\ &-2\begin{bmatrix}\phi_{R,B} & \phi_{I,B}\end{bmatrix}J\begin{bmatrix}\dot{\phi}_{R,B}\\ \dot{\phi}_{I,B}\end{bmatrix}\Bigg\}.\end{aligned} \tag{21}$$

The constraint is seen to enter at the lowest order possible (ϕ_B^4), because it involves a total derivative $\partial_\tau(\phi_{R,B}^2+\phi_{I,B}^2)$ that would vanish at order ϕ_B^2. Further, all possible combinations of ϕ_B and $\dot{\phi}_B$ at this order contribute, and only the relative coefficients specify this interaction as enforcing the Carnot constraint. Therefore, all the most relevant terms are represented to fourth order in ϕ_B.

Equations (20) and (21) show, formally, how modal properties should be summed in a partition function with the correct symmetries to represent an imperfect Carnot engine. However, the effective action superficially representing classical field correlations is not expressed directly in terms of the same bare mode amplitudes that define the microscopic theory. Rather, classical field properties are approximately represented in the action form, when it appears in a cutoff effective-field sum, with all frequency scales referenced to the experimental scale, including the high-frequency cutoff [13]. This precludes large integrals in interaction loops from invalidating the approximate equations of motion at tree level, as long as the coupling is small.

At leading order, the effective-field sum is approximated by introducing classically renormalized fields,

$$\begin{bmatrix}\phi_R\\ \phi_I\end{bmatrix}\equiv\sqrt{\frac{\omega_0}{\omega_R}}\begin{bmatrix}\phi_{R,B}\\ \phi_{I,B}\end{bmatrix}, \tag{22}$$

and the corresponding renormalized coupling $\overline{g}\equiv\overline{g_B}\omega_R/\omega_0$. These scale changes account for the dimension-determined scaling of corrections from large interaction integrals. Because they are power-law, anomalous scaling corrections should be subleading. Further, the renormalized coupling is smaller than the bare coupling, so if the effective theory is valid at any scale, it must remain so at smaller scales. Finally, the phenomenological description of the engine presented here is defined from symmmetries *a priori*. As long as the scaling to classical fields does not lead to large couplings or untreated breaking of assumed symmetries, anomalous corrections are not meaningful in defining the effective theory anyway, and so will be ignored. The existence of a valid, weak-coupling expansion, with all broken symmetries explicitly treated, will be taken after the fact as the demonstration that this preservation of the formal structure of the partition function, with classically normalized fields, is a plausible step.

It will be convenient, from this point onward, to represent S_E as a matrix trace. In terms of renormalized fields, the defining action then becomes

$$\frac{S_E}{\hbar}=\frac{1}{\omega_0}\oint\frac{d\tau}{2}\mathrm{Tr}\left\{\begin{bmatrix}\dot\phi_R\\ \dot\phi_I\end{bmatrix}[\dot\phi_R \quad \dot\phi_I]+\omega_0^2\begin{bmatrix}\phi_R\\ \phi_I\end{bmatrix}[\phi_R \quad \phi_I] + \bar g\left(\begin{bmatrix}c & -s\\ s & c\end{bmatrix}\begin{bmatrix}\dot\phi_R\\ \dot\phi_I\end{bmatrix}[\phi_R \quad \phi_I]\right)^2 - 2J\begin{bmatrix}\dot\phi_R\\ \dot\phi_I\end{bmatrix}[\phi_R \quad \phi_I]\right\}. \tag{23}$$

Setting $J=0$ in the notation (except when it is implicitly varied to equate expectation values of currents), and defining the renormalized constraint $\mathcal{C}\equiv(\omega_0/\omega_R)\mathcal{C}_B$, the Euclidean equation of motion from variation of Eq. (23) is

$$\left\{-\partial_\tau^2+\omega_0^2-2\bar g s\left(\mathcal{C}\begin{bmatrix} & 1\\ -1 & \end{bmatrix}\partial_\tau+\frac{\partial_\tau\mathcal{C}}{2s}\begin{bmatrix}c & s\\ -s & c\end{bmatrix}\right)\right\}\begin{bmatrix}\phi_R\\ \phi_I\end{bmatrix}=0. \tag{24}$$

Steady-state solutions, if such are found, have $\partial_\tau\mathcal{C}=0$ by definition. Further, the low-frequency dynamics of nonequilibrium solutions, such as the exponential growth away from a supercooled quiescent state, are driven by the coupling of order s to traveling waves, and produce $\partial_\tau\mathcal{C}\sim s\mathcal{C}$. Therefore, if the regulated constraint term in Eq. (23) is expected to lead to a uniform, linear-order perturbation to the free equations of motion (24), as results from the idealized δ functional, the coupling must scale as $\bar g|s|\to\bar g_0$ as $s\to 0$. This represents the physical observation that coupling to the stack remains a finite perturbation on TW behavior, even as the imposed driving gradient vanishes. The limit of free sound will arise from $\bar g_0\to 0$ at any fixed s. Though finite $\bar g_0$ requires $\bar g\to\infty$ at $s\to 0$, this scaling will be shown to lead to regular, sensible limits for all Green's functions in the MFT calculation. (This scaling will also be motivated by a modal decomposition of the constraint in Appendix A.)

It is worth noting that, in steady state, $\mathcal{C}$ is the *only* correction to the free Euclidean equations of motion. Since any interaction term $\mathcal{F}C$ other than $\mathcal{C}^2$, in Eq. (21), would give other corrections from variation of the factor $\mathcal{F}$, it follows that Eq. (23) is the most general functional enforcing the Carnot constraint and nothing else. Alternatively, it is the most general functional consistent with the maximal set of symmetries defining the engine, and thus must be the desired phenomenological Lagrangian.

The background field structure of Eq. (23) is probed by performing a Hubbard-Stratonovitch transformation to remove the quartic interaction term. An auxiliary field Q is introduced through a normalized Gaussian integral

$$1=\int\mathcal{D}Q\, e^{-S_{\mathrm{aux}}/\hbar}, \tag{25}$$

with

$$\frac{S_{\mathrm{aux}}}{\hbar}=\frac{1}{\omega_0}\oint\frac{d\tau}{2}\mathrm{Tr}\left\{\left(Q-i\sqrt{\bar g}\begin{bmatrix}c & -s\\ s & c\end{bmatrix}\begin{bmatrix}\dot\phi_R\\ \dot\phi_I\end{bmatrix}[\phi_R \quad \phi_I]\right)^2\right\}. \tag{26}$$

Inserting Eq. (25) into the partition function (20) gives a sum action $S_{E'}=S_E+S_{\mathrm{aux}}$ of the form

$$\frac{S_{E'}}{\hbar}=\frac{1}{\omega_0}\oint\frac{d\tau}{2}\mathrm{Tr}\left\{\begin{bmatrix}\dot\phi_R\\ \dot\phi_I\end{bmatrix}[\dot\phi_R \quad \dot\phi_I]+\omega_0^2\begin{bmatrix}\phi_R\\ \phi_I\end{bmatrix}[\phi_R \quad \phi_I] - 2i\sqrt{\bar g}Q\begin{bmatrix}c & -s\\ s & c\end{bmatrix}\begin{bmatrix}\dot\phi_R\\ \dot\phi_I\end{bmatrix}[\phi_R \quad \phi_I]+Q^2\right\}. \tag{27}$$

Variation with respect to J about $J=0$ in Eq. (23), and shift of the auxiliary field of integration, gives

$$i\sqrt{\bar g}\begin{bmatrix}c & -s\\ s & c\end{bmatrix}\left\langle\begin{bmatrix}\dot\phi_R\\ \dot\phi_I\end{bmatrix}[\phi_R \quad \phi_I]\right\rangle=\langle Q\rangle\equiv Q_0, \tag{28}$$

which defines the background field Q_0. Weak-coupling MFT consists of splitting $Q=Q_0+Q'$, showing that fluctuations Q' may be ignored, and solving self-consistently for Q_0 through its effect on ϕ Green's functions.

IV. STATIONARY POINTS AND FLUCTUATIONS

Solutions for Q_0 will be found by assuming a given form, showing that it is consistent with the existence of stationary points, and then using the symmetries of S_E' to show that all allowed solutions have the proposed form. It is shown in Appendix A that all constant Q_0 thus found couple only to zero-frequency constraint components from the TW sector, so differences between the use of $\mathcal{C}$ and $\bar{\mathcal{C}}$ in Eq. (23) are invisible in MFT.

Stationary points will be assumed to lie in the same SO(2) subgroup of SU(2) as the constraint matrix in Eq. (14):

$$Q_0=q\begin{bmatrix}c' & s'\\ -s' & c'\end{bmatrix}, \tag{29}$$

with the definitions $c'\equiv\cos\zeta$, $s'\equiv\sin\zeta$. The ϕ Green's function in Eq. (28) then takes the form

$$\left\langle \begin{bmatrix} \dot{\phi}_R \\ \dot{\phi}_I \end{bmatrix} [\phi_R \quad \phi_I] \right\rangle = \frac{-iq}{\sqrt{\bar{g}}} \begin{bmatrix} \cos(\zeta+\chi) & \sin(\zeta+\chi) \\ -\sin(\zeta+\chi) & \cos(\zeta+\chi) \end{bmatrix}. \tag{30}$$

To solve the self-consistency condition, one introduces the mode expansion

$$\begin{bmatrix} \phi_R(\tau) \\ \phi_I(\tau) \end{bmatrix} = \left(\omega_0 \oint d\tau \right)^{-1/2} \sum_n \begin{bmatrix} \phi_R \\ \phi_I \end{bmatrix}_n e^{i\omega_n \tau}, \tag{31}$$

where $\omega_n \equiv 2\pi n / \oint d\tau$ for bosonic Euclidean fields.

The modal solution to Eq. (28) is then

$$q \begin{bmatrix} c' & s' \\ -s' & c' \end{bmatrix} = \frac{-\sqrt{\bar{g}}\omega_0}{\oint d\tau} \sum_n \frac{\omega_n(\omega_n^2+\omega_0^2) \begin{bmatrix} c & -s \\ s & c \end{bmatrix} + 2\sqrt{\bar{g}}q\omega_n^2 \begin{bmatrix} c' & -s' \\ s' & c' \end{bmatrix}}{\text{Det}}, \tag{32}$$

with the denominator given by

$$\text{Det} \equiv [\omega_n^2 + \omega_0^2 + 2\sqrt{\bar{g}}q\omega_n \cos(\zeta-\chi)]^2 + [2\sqrt{\bar{g}}q\omega_n \sin(\zeta-\chi)]^2. \tag{33}$$

Independent matrix coefficients of Eq. (32) may be set equal as components of a vector:

$$q \begin{bmatrix} c' \\ s' \end{bmatrix} = -\sqrt{\bar{g}}\omega_0 \mathcal{G}_1 \begin{bmatrix} c \\ -s \end{bmatrix} - 2\bar{g}q\omega_0 \mathcal{G}_2 \begin{bmatrix} c' \\ -s' \end{bmatrix}. \tag{34}$$

The two independent modal Green's functions appearing in Eq. (34) are

$$\mathcal{G}_1 \equiv \frac{1}{\oint d\tau} \sum_n \frac{\omega_n(\omega_n^2+\omega_0^2)}{\text{Det}}, \tag{35}$$

$$\mathcal{G}_2 \equiv \frac{1}{\oint d\tau} \sum_n \frac{\omega_n^2}{\text{Det}}. \tag{36}$$

These are most conveniently evaluated in a low-temperature limit, where $\Sigma_n \rightarrow (\oint d\tau / 2\pi) \int d\omega$. Defining a condensed notation for square roots that arise from the determinant (33),

$$\sqrt{\pm} \equiv \sqrt{\omega_0^2 - \bar{g}q^2 \exp[\pm 2i(\zeta-\chi)]}, \tag{37}$$

and another for a combination of these that appears repeatedly,

$$a \equiv \frac{\sqrt{-} - \sqrt{+}}{\sqrt{-} + \sqrt{+}}, \tag{38}$$

the Green's functions evaluate to

$$\mathcal{G}_1 \rightarrow \frac{-\sqrt{\bar{g}}q}{4} \left(\frac{1}{\sqrt{+}} + \frac{1}{\sqrt{-}} \right) \left\{ 2\cos(\zeta-\chi) - ia \frac{\cos 2(\zeta-\chi)}{\sin(\zeta-\chi)} \right\}, \tag{39}$$

$$\mathcal{G}_2 \rightarrow \frac{1}{8} \left(\frac{1}{\sqrt{+}} + \frac{1}{\sqrt{-}} \right) \{1 - ia\cot(\zeta-\chi)\}. \tag{40}$$

After some algebra to simplify products of the various sines and cosines, Eqs. (39) and (40) in Eq. (34) reduce to the eigenvalue relation

$$\begin{bmatrix} c' \\ s' \end{bmatrix} = \frac{\bar{g}\omega_0 \cos 2\chi}{4} \left(\frac{1}{\sqrt{+}} + \frac{1}{\sqrt{-}} \right) (1 - ia\tan 2\chi) \left\{ \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} - \left(\frac{\tan 2\chi + ia}{1 - ia\tan 2\chi} \right) \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \right\} \begin{bmatrix} c' \\ s' \end{bmatrix}. \tag{41}$$

Simultaneous solution for ζ and q is easiest in the limits of large $|\bar{g}q^2|/\omega_0^2$, where the only self-consistent solutions have $|\sin 2(\zeta-\chi)| \ll 1$. Taylor expanding Eq. (38) to leading order in this small sine, irrespective of the root convention, gives

$$a \approx \frac{i\bar{g}q^2 \sin 2(\zeta-\chi)}{2[\omega_0^2 - \bar{g}q^2 \cos 2(\zeta-\chi)]}. \tag{42}$$

Eigenvectors of Eq. (41) with small $\zeta-\chi$ are only possible if $\tan 2\chi = -ia$, giving two solutions for $\zeta+\chi$ in terms of q:

$$\zeta+\chi \approx 2\chi \frac{\omega_0^2}{\bar{g}q^2} \quad \text{and} \quad \cos 2(\zeta-\chi) = 1 - O(\chi^2), \tag{43}$$

$$\zeta+\chi \approx \frac{\pi}{2}\,\text{sgn}(\chi q^2) - 2\chi \frac{\omega_0^2}{\bar{g}q^2} \quad \text{and} \quad \cos 2(\zeta-\chi) = -1 + \mathcal{O}(\chi^2). \tag{44}$$

To solve for q, one notes that $q^2 > 0$ leads to an imaginary expectation value for the Green's function (30) and a negative ϕ^4 interaction, while $q^2 < 0$ gives a real Green's function and positive interaction. (It may also be checked that the negative interaction is large, in maximal violation of the constraint, while the positive solution is small, in maximal compliance with it. The physical interpretation of these relations will be remarked upon below.)

Starting from real-valued fields ϕ_R, ϕ_I, the hypercontour of steepest descents in Eq. (20) may be displaced into the complex plane, but ϕ_R, ϕ_I remain single-component fields. Therefore, only $q^2>0$ stationary points are found. With the signs of $\cos 2(\zeta-\chi)$ from Eqs. (43) and (44), the unique real-q solutions at large or small g are then given by

$$\sqrt{\bar{g}}q \approx \omega_0 \sqrt{1-\bar{g}^2/4}; \quad \bar{g} \ll 2 \quad \text{and} \quad \cos 2(\zeta-\chi) \approx 1, \tag{45}$$

$$\sqrt{\bar{g}}q \approx \omega_0 \sqrt{\bar{g}^2/4-1}; \quad \bar{g} \gg 2 \quad \text{and} \quad \cos 2(\zeta-\chi) \approx -1. \tag{46}$$

The relative signs of square roots are determined, together with the angle, by the requirement that Green's functions continue through $\bar{g} \sim 2$ in the same quadrant of SO(2). The absolute sign of roots is then determined by causality after continuation back to real-time correlations. With the roots so chosen, the Euclidean Green's function (30) for $\bar{g} \ll 2$ is

$$\left\langle \begin{bmatrix} \dot{\phi}_R \\ \dot{\phi}_I \end{bmatrix} [\phi_R \quad \phi_I] \right\rangle \to -i\frac{\omega_0}{2}\left(\frac{2}{\bar{g}}\sqrt{1-\bar{g}^2/4} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} + \bar{g}_0 \,\mathrm{sgn}(\chi) \frac{(2/\bar{g})^2}{\sqrt{1-\bar{g}^2/4}} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \right), \tag{47}$$

while the solution for $\bar{g} \gg 2$ is

$$\left\langle \begin{bmatrix} \dot{\phi}_R \\ \dot{\phi}_I \end{bmatrix} [\phi_R \quad \phi_I] \right\rangle \to -i\frac{\omega_0}{2}\left(\bar{g}_0 \frac{(2/\bar{g})^3}{\sqrt{1-4/\bar{g}^2}} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} + \mathrm{sgn}(\chi)\sqrt{1-4/\bar{g}^2} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \right). \tag{48}$$

Rotation back to real-time currents, via $\partial_\tau \to i\partial_t$, gives the solution for $\bar{g} \ll 2$:

$$\left\langle \partial_t \begin{bmatrix} \phi_R \\ \phi_I \end{bmatrix} [\phi_R \quad \phi_I] \right\rangle \to \frac{\omega_0}{2}\left(\frac{2}{\bar{g}}\sqrt{1-\bar{g}^2/4} \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} + \bar{g}_0 \,\mathrm{sgn}(\chi) \frac{(2/\bar{g})^2}{\sqrt{1-\bar{g}^2/4}} \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \right), \tag{49}$$

and for $\bar{g} \gg 2$:

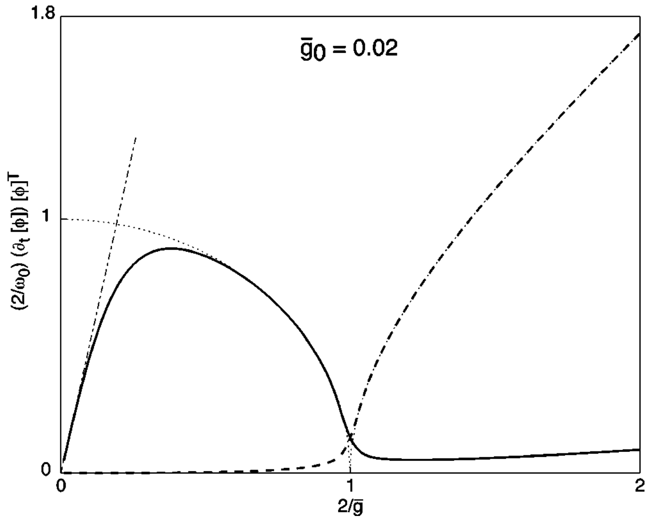

FIG. 2. Coefficients of independent matrix terms in the current expectation values. Solid black is the off-diagonal component (fluctuations included); dashed black is the diagonal component. Dotted lines are $\bar{g}_0 \to 0$ limiting values given in Eq. (51). Dash-dot line is the s-linear limit of the tanh function in Eq. (61).

$$\left\langle \partial_t \begin{bmatrix} \phi_R \\ \phi_I \end{bmatrix} [\phi_R \quad \phi_I] \right\rangle \to \frac{\omega_0}{2}\left(\bar{g}_0 \frac{(2/\bar{g})^3}{\sqrt{1-4/\bar{g}^2}} \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} + \mathrm{sgn}(\chi)\sqrt{1-4/\bar{g}^2} \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \right). \tag{50}$$

The importance of the way the positive-q^2 solution violates the sign preferred by the constraint may now be seen. The off-diagonal current in both Eqs. (49) and (50) has a sign corresponding to the growing solution under Eq. (11), but the diagonal magnitude decays thermally, as required for a causal solution. That the constraint mitigates this decay is seen by the decrease in magnitude of the diagonal term at large $\bar{g}$.

The parameter $\bar{g}$ selects in the same functional way between the two asymptotic solutions, at whatever value of $\bar{g}_0$. However, $\bar{g}_0$ remains as a dimensionless parameter determining the form of the solutions, and the continuation between them. At small $\bar{g}_0$, the transition at $\bar{g}=2$ resembles the Curie point in the classical Landau description of ferromagnetism [14], as seen in Fig. 2. Indeed, as $\bar{g}_0 \to 0$, the current Green's functions simplify to

$$\left\langle \partial_t \begin{bmatrix} \phi_R \\ \phi_I \end{bmatrix} [\phi_R \quad \phi_I] \right\rangle \to \frac{\omega_0}{2}\left(\frac{2}{\bar{g}}\sqrt{1-\bar{g}^2/4} \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} \right), \quad \bar{g} \leqslant 2$$
$$\to \frac{\omega_0}{2}\left(\mathrm{sgn}(\chi)\sqrt{1-4/\bar{g}^2} \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \right), \quad \bar{g} \geqslant 2. \tag{51}$$

However, at nonzero $\overline{g}_0$, this transition is always regular, and is not the critical point relevant to the TW onset transition. Rather, it quantifies the intuitive expectation that an insufficiently strongly coupled stack, relative to the driving gradient it is asked to impose, becomes unable to sustain coherent order.

The fact that the stationary points (49) and (50) remain finite and dependent on $\mathrm{sgn}(\chi)$ at $\chi \to 0$ ($\overline{g} \to \infty$), where nonzero solutions must be degenerate, motivates consideration of fluctuations about Q_0 at small χ. The degeneracy and completeness of the stationary points is most easily studied by momentarily promoting ϕ_R, ϕ_I to complex-valued fields, and replacing the current dyadic above with

$$\begin{bmatrix} \dot{\phi}_R \\ \dot{\phi}_I \end{bmatrix} [\phi_R \quad \phi_I] \to \begin{bmatrix} \dot{\phi}_R \\ \dot{\phi}_I \end{bmatrix} [\phi_R^* \quad \phi_I^*] \tag{52}$$

(and similarly with all other dyadics).

Given any stationary solution Q_0, a hypersurface of values in the Q integral may be formed as

$$Q = R Q_0 R^{\dagger}, \tag{53}$$

by acting with a rotation $R \in \mathrm{SU}(2)$. Decomposing general Q into the basis elements

$$Q = q^0 \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} + q^1 \begin{bmatrix} & i \\ -i & \end{bmatrix} + q^2 \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} + q^3 \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \tag{54}$$

the action of R in Eq. (53) may be written as

$$\begin{bmatrix} q^1 \\ q^2 \\ q^3 \end{bmatrix} = \mathcal{R} \begin{bmatrix} q_0^1 \\ q_0^2 \\ q_0^3 \end{bmatrix}, \tag{55}$$

for $\mathcal{R} \in \mathrm{SO}(3)$.

Meanwhile, a similar shift on fields ϕ, ϕ^* may be performed in the action (27), replacing

$$\begin{bmatrix} \dot{\phi}_R \\ \dot{\phi}_I \end{bmatrix} [\phi_R^* \quad \phi_I^*] \to R \begin{bmatrix} \dot{\phi}_R \\ \dot{\phi}_I \end{bmatrix} [\phi_R^* \quad \phi_I^*] R^{\dagger} + O(\dot{R}), \tag{56}$$

and likewise with other dyadics. For constant R, this shift of ϕ fields is a symmetry of the measure, and at $s=0$, for each zero-frequency fluctuation of Q in Eq. (53), it may be performed to yield an exact symmetry of the action (27). Therefore, the complete spectrum of stationary points at $s=0$ is the image of Q_0 found above under $\mathrm{SO}(3)$. Nonzero s breaks this degeneracy, and if terms $O(\dot{R})$ are ignored in Eq. (56) at low temperature, the remaining terms give the effective potential for R.

Returning now to the simpler case ϕ_R, ϕ_I real, $\mathrm{SO}(3)$ breaks to $\mathrm{SO}(2) \times Z_2$. The $\mathrm{SO}(2)$ factor comes from $R^{\dagger} = R^T$, and leaves Q_0 invariant. (This is just the global symmetry whose local form gives the Goldstone sector, because rotation of traveling waves in the spatial plane is equivalent to offsetting the zero of time.) The residual Z_2 comes from $\pm \pi/2$ rotations in either of the remaining $\mathrm{SU}(2)$ generators. It is a symmetry of the action under the discrete transformations $\phi_R \leftrightarrow \phi_I$, and $\phi_R \to \phi_R$, $\phi_I \to -\phi_I$ for the two generators, respectively. Thus for real fields, an integral over fluctuations reduces to the discrete sum over antipodal stationary points at $s \sim 0$.

Letting 2θ denote the elevation angle in $\mathrm{SO}(3)$, or $\cos(2\theta) = \pm 1$ in Z_2, the effective potential may be estimated by inserting the mean current Green's function (30) in the action (27), to yield

$$\frac{S'_E}{\hbar}[\theta; s] = \frac{S'_E}{\hbar}\Big|_{s=0} - \frac{\overline{g} s^2 \omega_0}{2} \oint d\tau [1 + \cos(2\theta(\tau))]. \tag{57}$$

Because only the surface of fluctuations (53) is degenerate at $s=0$, other modes are massive and can be ignored in evaluating the expectation value,

$$\langle Q \rangle = \frac{\int \mathcal{D}R\, e^{\overline{g} s^2 \omega_0 \oint d\tau \cos(2\theta)/2} q \cos(2\theta)}{\int \mathcal{D}R e^{\overline{g} s^2 \omega_0 \oint d\tau \cos(2\theta)/2}} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}. \tag{58}$$

For complex fields and $\mathcal{R}$ in $\mathrm{SO}(3)$, the measure would be

$$\int \mathcal{D}R = 2\pi \int_{-1}^{1} d\cos(2\theta), \tag{59}$$

while for real fields and $\mathcal{R}$ in Z_2, it is simply

$$\int \mathcal{D}R = \sum_{\cos(2\theta)=-1}^{1}. \tag{60}$$

The evaluation of Eq. (58) with measure (60) is

$$\langle Q \rangle = \frac{\sqrt{\overline{g}}\omega_0}{2} \mathrm{sgn}(\chi) \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \tanh\left(\frac{\overline{g} s^2 \omega_0}{2} \oint d\tau \right) \tag{61}$$

[the $\mathrm{SO}(3)$ case differs by a prefactor and higher-order terms]. Recognizing that $\overline{g} s^2 \,\mathrm{sgn}(\chi) = \overline{g}_0 \chi + O(\chi^3)$, the current Green's function to leading order in small χ becomes

$$\frac{1}{\omega_0} \left\langle \partial_t \begin{bmatrix} \phi_R \\ \phi_I \end{bmatrix} [\phi_R \quad \phi_I] \right\rangle \to -\frac{\overline{g}_0 \chi \omega_0 \oint d\tau}{4} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}. \tag{62}$$

The auxiliary field Q does not directly correspond to individual backgrounds ϕ_R, ϕ_I, because $\mathrm{SO}(2)$ rotations that phase-shift traveling waves leave Q_0 invariant. However, because the leading behavior of the engine is still constrained by the dynamics of free phonons, the interpretation of Eq. (62) in terms of classical real-time backgrounds is unambiguous. Finite TW currents come from allowed solutions of the form

$$\begin{bmatrix} \phi_R \\ \phi_I \end{bmatrix} = A \begin{bmatrix} \cos(\omega_0 t) \\ \sin(\omega_0 t) \end{bmatrix}, \tag{63}$$

up to choice of the zero of time. Matching the current expectation value from such a classical background to Eq. (62), and recalling that $\oint d\tau = \hbar/k_B T$, gives the scaling for $\chi > 0$,

$$A \approx \sqrt{\frac{\overline{g}_0 \chi}{2}} \sqrt{\frac{\hbar \omega_0}{k_B T}}. \tag{64}$$

V. DISCUSSION

Equation (64) predicts a universal scaling of the saturation amplitude with driving gradient χ near the TW critical point, in the sense that it follows from the most relevant terms in an effective field theory of free phonons, perturbed by imperfect Carnot self-amplification. The dependence of A on χ is manifestly nonanalytic at $\chi = 0$. Apart from the fact that the critical coupling is zero, this scaling is the same as that of the averaged magnetization of a ferromagnet in the Ginzburg-Landau treatment [14]: $A \sim \sqrt{\chi}$ for $\chi > 0$ and $A \equiv 0$ otherwise. The amplitude of the counterpropagating wave has exactly the dual behavior, with respect to $-\chi$.

The current (proportional to A^2), which couples directly to χ, is analytic through $\chi = 0$, so the argument that $\overline{g}_0$ approaches a constant there implies saturation proportional to the classical gain times a fixed coupling strength. This may be in agreement with Ref. [2], but those data were not presented to test this point.

Unfortunately, because even the scaling of the saturation amplitude near the critical point is difficult to measure, it is not clear how to assign more directly an experimental value to the effective coupling $\overline{g}_0$. In the phenomenological equations at the level of the local ideal gas, irreversibility is encoded in conductivities that damp wave solutions or parasitically reduce gain. In the reversible action (23), the only way these can be encoded is by weakening the Carnot constraint, so that not all energy extracted from the reservoirs is represented by growth of sound in the engine. The relation of this phenomenological constraint to the underlying ideal gas conductivities appears as difficult to derive as to measure directly.

It is appealing to speculate that, in spite of its formal relation to a ''bare'' coupling by a rescaling involving ω_0, $\overline{g}_0$ is at most a function of stack properties and temperature. In that case, since ω_0 can be varied independently by varying resonator length L, the saturation current would be proportional to $\hbar \omega_0 / kT$, the ratio of the number of driving engine cycles per unit time, to the thermal decay rate. It may also be noted that at $\overline{g}_0 \to 0$, fluctuations suppress $\langle Q \rangle$ at all s, and the free theory is recovered.

If the scaling regime with ΔT predicted by Eq. (64) could be found experimentally, the effective coupling would then be determined by the coefficient as a function of T. A difficulty with this is that experiments are only currently feasible on the SW critical point, as in Ref. [2], and the symmetry-based calculations above are too limited to show whether the TW critical point should have the same scaling. It is therefore of interest experimentally to pursue engines that can drive traveling waves, and theoretically, to extend the phase-transition interpretation from the relatively natural reversible case, to include the SW cycle as well.

A final comment concerns the relation of the background current (62) to the selection of a definite phase for classical backgrounds (63). In a ferromagnetic spin system, there are a large number of microscopic spins, which need not be phase-coherent to produce a background magnetization. Under coarse-graining, they are replaced with a classical magnetization vector, which is independent of those phases. In the acoustic resonator, the current is still constructed explicitly from wave solutions, but there is only one ''spin.'' Possible ground states all have expectation values which are linear combinations of the two independent temporal modes (differing by the $\pi/2$ phase), and any such state may be rotated to the form (63) by appropriate choice of the zero of time. Therefore, formation of a background current requires a sum over independent ground states with definite phases. In classical correlations, these are engine cycles that spontaneously break time-translation symmetry, and create long-range dynamical correlations with the field configurations at any one time.

VI. CONCLUSIONS

The foregoing derivation took as input a set of effective actions from Ref. [8], which have already been shown to lead to an intriguing connection between Carnot's theorem and the analytic structure derived from finite-temperature summation of classical configurations. Through explicit formulation, and then coarse-graining, of the sum, the same actions and analyticity have been shown to lead to spontaneous symmetry breaking and finite-amplitude saturation of the driven sound mode.

The notion that finite-temperature disorder could select among, and then stabilize, such classical configurations, would be implausible, except that exact degeneracy of orthogonal engine modes is strictly enforced by time-translation invariance of the underlying dynamics. In the equilibrium representation, phase drift among these modes would be the Goldstone excitation of the theory. It appears that the mechanism by which thermal disorder causes finite-amplitude saturation is diffusion of the work extracted from existing cycles, over the most accessible states, which should be visible as phase meandering dynamically [15]. Near the critical point, the saturation of such phase noise should be the mechanism that ''melts'' the dynamical long-range order of a coherent engine cycle. The measurement and prediction of the spectra of phase noise in close neighborhoods of onset are therefore important directions for experimental and theoretical future work.

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APPENDIX A: THE AVERAGED CONSTRAINT AND SCALING OF THE COUPLING

The idealized δ functional (15) is linear in the constraint it enforces. When the regulators g_B, ω are introduced in Eq. (17), this linear functional is replaced with a quadratic interaction term $S_E^{\mathrm{int}}/\hbar$, in Eq. (18). In order for the regulator to enforce vanishing of the low-frequency components of $\mathcal{C}_B$ uniformly as $s\to 0$, it should effectively lead to a linear constraint term with s-invariant weight. A mode expansion of the interaction will be used here to show that this physical condition requires scaling of the assumed ''bare'' coupling with s. The result will then be shown to admit substitution of the local constraint for the derivation of mean-field backgrounds, if the effective coupling is corrected by an additional scale factor.

The derivation is carried out in the renormalized fields of Eq. (22), and the corresponding constraint $\mathcal{C}$ of Eq. (24), to match the bulk of the main text. Translation from the bare quantities of Sec. III is by simple rescaling. It is also convenient at this point to denote the rotation matrix

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \equiv R_s .$$

Only ϕ bilinears, near-diagonal in a TW basis, contribute to the time-averaged constraint $\overline{\mathcal{C}}$, so it is useful to expand ϕ in TW mode coefficients,

$$\begin{bmatrix} \phi_R(\tau) \\ \phi_I(\tau) \end{bmatrix} = \left(\omega_0 \oint d\tau \right)^{-1/2} \sum_n \left(\frac{\phi_n}{2} \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{i\omega_n \tau} + \frac{\phi_n^*}{2} \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-i\omega_n \tau} \right), \tag{A1}$$

in favor of the generic expansion (31). Positive ω_n represent analytic traveling waves, and negative ω_n are antianalytic. Modes of $\mathcal{C}$ will be normalized as in Eqs. (31) and (A1):

$$C(\tau) = \left(\omega_0 \oint d\tau \right)^{-1/2} \sum_k C_k e^{i\omega_k \tau}. \tag{A2}$$

With these definitions,

$$C_k = \left(\omega_0 \oint d\tau \right)^{-1/2} \sum_n \phi_n \phi_{n-k}^* \left(\frac{i\omega_k}{4} e^{-i\chi} - \frac{\omega_n s}{2} \right). \tag{A3}$$

The modes of the local and time-averaged constraints relate as $\overline{\mathcal{C}_k} \approx \mathcal{C}_k$ for $|\omega_k| \ll \overline{\omega} \sim \omega_0/Q$ and $\overline{\mathcal{C}_k} \approx 0$ for $|\omega_k| \geq \overline{\omega}$. Scaling of bilinears in ϕ is determined at leading order by the free finite-temperature theory, so the magnitude of $\overline{\mathcal{C}_k}$ is determined by the two frequency coefficients that appear in Eq. (A3). The left-hand term in parentheses scales as $|\omega_k| \leq \overline{\omega} \sim \omega_0 s$, and the right-hand term scales explicitly as $\omega_n s$. Therefore, $\overline{\mathcal{C}}$ is $O(s)$, and the coefficient of $\overline{\mathcal{C}_k}$ in the mode expansion of $S_E^{\text{int}}/\hbar$,

$$\frac{(g_B \omega_R/\omega_0)}{2\omega_0} \oint d\tau \overline{\mathcal{C}}^2 = \frac{1}{2\omega_0^2} \sum_k \left(\frac{g_B \omega_R \overline{\mathcal{C}_{-k}}}{\omega_0} \right) \overline{\mathcal{C}_k}, \tag{A4}$$

remains nontrivial and uniform as $s\to 0$, only if $g_B|s| \to g_{B,0}$.

Though this scaling appears singular (especially when applied to the local constraint for MFT calculations), as long as $g_{B,0}$ remains small, the weak-coupling expansion is valid for all interacting fluctuations in the original definition. The strongly coupled interactions would all be standing waves and counter propagating traveling waves, which are not present in the time-averaged constraint.

To see that the assumption of the averaged constraint in the definitions is consistent with use of the local constraint in MFT calculations, it is then convenient to return to the mode expansion (31), in terms of which

$$\mathcal{C}_k = \left(\omega_0 \oint d\tau \right)^{-1/2} \mathrm{Tr} \left\{ R_s \sum_n i\omega_n \begin{bmatrix} \phi_R \\ \phi_I \end{bmatrix}_n [\phi_R \quad \phi_I]_{-n+k} \right\}. \tag{A5}$$

Even though it is not time local, the constraint action (A4) can be canceled explicitly by defining a two-index auxiliary field,

$$\tilde{Q}_{n,n'-k} \equiv Q_{n,n'-k} + \sqrt{\frac{g_B \omega_R/\omega_0}{\omega_0 \oint d\tau}} R_s \omega_n \begin{bmatrix} \phi_R \\ \phi_I \end{bmatrix}_n [\phi_R \quad \phi_I]_{-n'+k}, \tag{A6}$$

and replacing the action (26) with the expansion

$$\frac{S_{\text{aux}}}{\hbar} = \frac{1}{2\omega_0^2} \sum_{n\,n'} \sum_{k \text{ small}} \mathrm{Tr}\{ \tilde{Q}_{n,n'-k} \tilde{Q}_{n',n+k} \}, \tag{A7}$$

where finite $\overline{\omega}$ has been represented by truncating the range of k summation. The sum of interaction and auxiliary field actions is then

$$\frac{S_{\text{aux}} + S_E^{\text{int}}}{\hbar} = \frac{1}{2\omega_0^2} \sum_{k \text{ small}} \sum_{n'\,n} \mathrm{Tr} \times \left\{ Q_{n,n'-k} Q_{n',n+k} + 2\sqrt{\frac{g_B \omega_R/\omega_0}{\omega_0 \oint d\tau}} Q_{n',n+k} R_s \omega_n \times \begin{bmatrix} \phi_R \\ \phi_I \end{bmatrix}_n [\phi_R \quad \phi_I]_{k-n'} \right\}. \tag{A8}$$

Equation (A7) cannot represent any product of time-local fields, as it must cancel a product of time-averaged constraints. However, assuming a time-independent mean-field background remains consistent, and may be applied to modes (A6) as

$$Q_{n,n'-k} = \left(\frac{\omega_0 \oint d\tau}{\sum_{k \text{ small}} \sum_n} \right)^{1/2} Q_0 \delta_{n,n'-k} + Q'_{n,n'-k}. \tag{A9}$$

To zero order in fluctuations, the terms of Eq. (A8) that survive are

$$\frac{S_{\rm aux}+S_E^{\rm int}}{\hbar}=\frac{1}{2\omega_0^2}{\rm Tr}\left\{\left(\omega_0\oint d\tau\right)Q_0^2+2\sqrt{\bar{g}}Q_0R_s\sum_n\omega_n\begin{bmatrix}\phi_R\\ \phi_I\end{bmatrix}_n[\phi_R\quad\phi_I]_{-n}\right\}+O(Q'), \tag{A10}$$

where the sum on n is *unrestricted*, but a new coupling has been defined to absorb a redundant sum on k:

$$\bar{g}\equiv g_B\frac{\omega_R}{\omega_0}\frac{\sum_{k\ \rm small}}{\sum_n}\approx g_B\frac{\omega_R}{\omega_0}\frac{\bar{\omega}}{\Lambda}, \tag{A11}$$

where Λ is the frequency cutoff defining the effective theory. If the local constraint had been assumed from the beginning, by raising $\bar{\omega}$ so that $\Sigma_{k\ \rm small}\to\Sigma_n$, $\bar{g}$ would manifestly return to the classically renormalized, bare coupling. Removing the classical scale factor to identify the coupling $\bar{g}_B=\bar{g}\omega_0/\omega_R$ that appears in Eq. (21), one recovers $\bar{g}_B=g_B\bar{\omega}/\Lambda$.

The relation to the time-local Hubbard-Stratonovitch field of Sec. III may be seen by taking the mode expansion

$$Q(\tau)=\left(\omega_0\oint d\tau\right)^{-1/2}\sum_k Q_k e^{i\omega_k\tau}\equiv Q_0+Q'(\tau). \tag{A12}$$

Starting from the expansion for the local interaction term, but with a rescaled coupling,

$$\begin{aligned}\frac{\bar{g}}{2\omega_0}\oint d\tau\left([\phi_R\quad\phi_I]R_s\begin{bmatrix}\dot{\phi}_R\\ \dot{\phi}_I\end{bmatrix}\right)^2&=\frac{\bar{g}}{2\omega_0^3\oint d\tau}\sum_{m,n,k'}\left(i\omega_n[\phi_R\quad\phi_I]_{k'-n}R_s\begin{bmatrix}\phi_R\\ \phi_I\end{bmatrix}_n\right)\\&\quad\times\left(i\omega_m[\phi_R\quad\phi_I]_{-k'-m}R_s\begin{bmatrix}\phi_R\\ \phi_I\end{bmatrix}_m\right),\end{aligned} \tag{A13}$$

shifting indices $k=n-m-k'$ to produce the tensor form,

$$\begin{aligned}\frac{\bar{g}}{2\omega_0}\oint d\tau\left([\phi_R\quad\phi_I]R_s\begin{bmatrix}\dot{\phi}_R\\ \dot{\phi}_I\end{bmatrix}\right)^2&=\frac{\bar{g}}{2\omega_0^3\oint d\tau}\sum_{m,n,k}{\rm Tr}\left\{\left(i\omega_nR_s\begin{bmatrix}\phi_R\\ \phi_I\end{bmatrix}_n[\phi_R\quad\phi_I]_{k-n}\right)\right.\\&\quad\left.\times\left(i\omega_mR_s\begin{bmatrix}\phi_R\\ \phi_I\end{bmatrix}_m[\phi_R\quad\phi_I]_{-k-m}\right)\right\},\end{aligned} \tag{A14}$$

and offsetting a time-local Q by all modes of ϕ as the modes of the field appearing squared in Eq. (26),

$$\tilde{Q}_k\equiv Q_k+\sqrt{\frac{\bar{g}}{\omega_0\oint d\tau}}R_s\sum_m\left(\omega_m\begin{bmatrix}\phi_R\\ \phi_I\end{bmatrix}_m[\phi_R\quad\phi_I]_{k-m}\right), \tag{A15}$$

gives the equation corresponding to Eq. (A8):

$$\frac{S_{\rm aux}+S_E^{\rm int}}{\hbar}=\frac{1}{2\omega_0^2}\sum_k{\rm Tr}\left\{Q_kQ_{-k}+2\sqrt{\frac{\bar{g}}{\omega_0\oint d\tau}}Q_kR_s\sum_n\omega_n\begin{bmatrix}\phi_R\\ \phi_I\end{bmatrix}_n[\phi_R\quad\phi_I]_{k-n}\right\}. \tag{A16}$$

The MFT condition (A9) for the constant mode gives the relative normalization $Q_k=(\omega_0\oint d\tau)^{1/2}Q_0\delta_{k0}+Q'_k$, leading again to Eq. (A10). This sequence of mode expansions corresponds to the manipulations of the everywhere-local action (21) carried out in Sec. III. Though it is not compatible *as a definition* with the scaling of $\bar{g}$ required to model the TW engine, the resulting MFT is the same as that obtained from the valid weak-coupling expansion induced by $\bar{\mathcal{C}}$.

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